

Complete subvarieties in moduli spaces of rank 2 stable sheaves on smooth projective curves and surfaces

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Running title

Complete subvarieties in moduli spaces

Abstract The aim of this paper is to prove the existence of large complete subvarieties in moduli spaces of rank two stable sheaves with arbitrary c_1 and sufficiently large c_2 , on algebraic surfaces. Then we study the restriction of these sheaves to curves of high degree embedded in the surface. In the final section we give a relation with the spin strata defined by Pidstrigach and Tyurin.

1 Introduction

Moduli spaces of rank 2 stable vector bundles over algebraic surfaces had attracted considerable interest in the last years in relation with the powerfull Donaldson's invariants. Apart their detailed structure, a natural problem is to find a "reasonable" bound for c_2 from which the moduli spaces are nonempty. The answer of this question has been given by many authors in different degrees of generality in [1],[5],[8].

A second question related more closely to the Donaldson's invariants is to find, at last in some special cases, complete subvarieties of moduli spaces, which can in principle be used for the calculus of Donaldson intersection numbers. For example in [2], [4] the authors construct complete curves in moduli spaces of stable bundles with rank 2 and c_2 sufficiently great.

The aim of this paper is to present a construction of complete subvarieties with large dimension (of order $2c_2$), in moduli spaces of rank 2 stables sheaves with c_2 sufficiently great, extending the basic construction of [1].

Then we compare the dimension of our subvarieties with the O'Grady bound from [6] and we study the restriction of the sheaves in our subvarieties to curves of high degree embedded in the surface, obtaining also complete subvarieties in moduli spaces of stables rank 2 bundles over curves. In the final section we give the relation of ours subvarieties with the spin strata defined in [10], [7] for the spin-polynomial invariants.

2 Construction of subvarieties

For fixing the notations let S be a smooth projective surface, H a very ample polarisation and $L \in \text{Pic}(S)$ a line bundle on S . Let $\widehat{M}_H(2, L, c_2)$ be the moduli space of H -stable rank 2 torsion free sheaves E with $\det(E) = L$ and $c_2(E) = c_2$. Let n_L and $\beta(L, H)$ be defined as follows: for $L \cdot H > 0$

$$\begin{aligned} n_L &= \inf \{n \in \mathbf{N}^* \mid nH^2 > \frac{L \cdot H}{2}, nH^2 > (L + K_S) \cdot H\} \\ \alpha(L, H) &= \max \{h^0(\mathcal{O}_S(L + K_S)) + 1, 2 + n_L H \cdot (L + K_S), 1 + n_L \frac{L \cdot H}{2}\}, \end{aligned}$$

and for arbitrary $L \cdot H$:

$$\beta(L, H) = \begin{cases} \alpha(L, H) & \text{if } L \cdot H > 0 \\ \alpha(-L, H) & \text{if } L \cdot H < 0 \\ \alpha(L + 2H, H) - H^2 & \text{if } L \cdot H = 0 \end{cases}$$

In [1] it was proven that the following theorem holds:

Theorem 2.1. *For all $c \geq \beta(L, H)$ there exists a rank two vector bundle H -stable with $c_1(E) = L$ and $c_2(E) = c$.*

In the present note we are concerned with the following result:

Theorem 2.2. *For c_2 sufficiently great (explicitly given), the moduli space $\widehat{M}_H(2, L, c_2)$ contains a smooth complete subvariety of dimension*

$$2c_2 + h^0(-L) - \chi(-L) - 1$$

which has the structure of a projective fiber bundles over symmetric product of a curve in S .

The expression "sufficiently great" means that c_2 must satisfy the following conditions:

$$c_2 \geq \beta(L, H)$$

$$c_2 > n_L \cdot L \cdot H$$

$$c_2 \geq 2g(C) - 1$$

where $C \in |n_L H|$ is smooth.

Proof:

As in the proof of the **Theorem 2.1.** in [1], the principal case is $L \cdot H > 0$; the other situations are reduced to this by dualising if $L \cdot H < 0$ or by taking the tensor product with $\mathcal{O}_S(H)$ if $L \cdot H = 0$. For simplicity we assume in the sequel that $L \cdot H > 0$.

First of all, for any smooth curve C on S and any $Z \in \text{Div}^{c_2}(C)$, Z can be seen in a single way as a 0-dimensional subscheme in C and through the embedding $C \subset S$ as a 0-dimensional local complete intersection subscheme on S . So we have a closed embedding of $\text{Div}^{c_2}(C) = \text{Sym}^{c_2}(C)$ in $\text{Hilb}^{c_2}(S)$ -the Hilbert scheme of 0-dimensional subscheme of length c_2 in S .

The main idea is that for a smooth curve $C \in |n_L H|$ and arbitrary $Z \in \text{Div}^{c_2}(C)$, any extension:

$$0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow \mathcal{O}_S(L) \otimes \mathcal{J}_Z \rightarrow 0$$

is H -stable and the generic extension is locally free. The H -stability is in fact a consequence of the basic lemma in [1] and the generic locally freeness follows from the general theory in [3].

Also, for $c_2 > n_L \cdot L \cdot H$ there are no two of them isomorphic because the existence of an isomorphism between the E 's of two different extension would imply that we have a diagram of the following type:

$$\begin{array}{ccccccc}
& & & O & & & \\
& & & \uparrow & & & \\
& & & \mathcal{O}_S(L)J_{Z'} & & & \\
& & & \uparrow & & & \\
O & \longrightarrow & \mathcal{O}_S & \longrightarrow & E & \longrightarrow & \mathcal{O}_S(L)J_Z \longrightarrow 0 \\
& & & & \uparrow & \nearrow \psi & \\
& & & & \mathcal{O}_S & & \\
& & & & \uparrow & & \\
& & & & 0 & &
\end{array}$$

The arrow ψ dose not vanish because otherwise the two extensions would be the same. This show that L has a section vanishing on Z and this contradict:

$$c_2 > n_L \cdot L \cdot H .$$

These extensions are parametrised by a projective fiber bundle over $Sym^{c_2}(C)$; the fiber over Z is the projectivisation of

$$\text{Ext}^1(LJ_Z, \mathcal{O}_S) \cong H^1(S, K_S LJ_Z)^*$$

The dimension of this bundle is easily calculated using the cohomology sequence of the standard sequence:

$$0 \rightarrow \mathcal{O}_S(K+L) \otimes \mathcal{J}_Z \rightarrow \mathcal{O}_S(K+L) \rightarrow \mathcal{O}_Z \rightarrow 0$$

In cohomology we have the following:

$$\begin{aligned}
0 \rightarrow H^0(\mathcal{O}_S(K+L) \otimes \mathcal{J}_Z) &\rightarrow H^0(\mathcal{O}_S(K+L)) \rightarrow H^0(\mathcal{O}_Z) \rightarrow \\
&\rightarrow H^1(\mathcal{O}_S(K+L) \otimes \mathcal{J}_Z) \rightarrow H^1(\mathcal{O}_S(K+L)) \rightarrow 0
\end{aligned}$$

So we have $\dim H^1(S, K_S LJ_Z) = c_2 + h^1(-L) - h^2(-L) = c_2 + h^0(-L) - \chi(-L)$ and consequently the projective bundle we have constructed has the fiber of dimension

$$c_2 + h^0(-L) - \chi(-L) - 1$$

and the complete subvariety has the dimension announced in **Theorem 2.2**.

Furthermore, if $c_2 \geq 2g(C) - 1$ then every line bundle over C of degree c_2 is nonspecial and therefore $Sym^{c_2}(C)$ is itself a smooth projective fiber bundle over the jacobian $J(C)$ of C , with the fiber of dimension $c_2 - g(C)$.

Finally, let us observe that if the line bundle L is nef and big, then the term $h^0(-L)$ vanish and so the dimension of our subvarieties depends only on numerical variables.

3 Comparison with the O'Grady bound

As we have seen in the preceeding section, the subvarieties we have constructed contain sheaves which are not locally free. The aim of this section is to relate the dimension of our subvarieties with a result obtained by O'Grady in [6] concerning the maximal dimension of complete subvarieties in moduli spaces of stables vector bundles.

For fixing the notations let E a rank two sheaf over S . Let

$$\Delta_E = c_2 - c_1^2/4$$

$$\Delta_0(H) = \begin{cases} 3H^2 & \text{if } K \cdot H < 0 \\ 3H^2(1 + \frac{K \cdot H}{H^2})^2 & \text{if } K \cdot H \geq 0 \end{cases}$$

$$\lambda_2 = \frac{23}{6}$$

$$\lambda_1(H) = \frac{1}{2\sqrt{3H^2}} \cdot (4H^2 + 3K \cdot H + 4)$$

Finally let $\lambda_0(H) :=$

$$\frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 4 - 3\chi(\mathcal{O}_S), \text{ if } K \cdot H < 0$$

and

$$\frac{3(K \cdot H)^2}{H^2} + 6K \cdot H + \frac{3H^2}{2} - \frac{K^2}{4} + 8 - 3\chi(\mathcal{O}_S), \text{ if } K \cdot H \geq 0.$$

Let \mathcal{M} be the moduli space of rank two H -semistables sheaves with c_2 and the determinant fixed. The theorem of O'Grady we are concerned with, is the following:

Theorem 3.1. *If H is very ample, $\Delta_E > \Delta_0(H)$ and \mathcal{V} is a complete subvariety of \mathcal{M} such that:*

$$\dim \mathcal{V} > \lambda_2 \cdot \Delta_E + \lambda_1(H) \cdot \sqrt{\Delta_E} + \lambda_0(H),$$

then \mathcal{V} contain sheaves which are not locally free.

A very simple computation shows that, at least in the limit $c_2 \gg 0$, the subvarieties constructed in **Section 2** have the dimension less than the bound in O'Grady theorem. So we can expect that there are complete subvarieties of the same dimension, as of ours, in which all the sheaves are locally free and H -stable. Unfortunately we can not provide an explicit construction for such (potentially existing) subvarieties.

4 Application to moduli spaces over curves

Let D be a smooth projective curve of genus g over the complex field. The moduli space of rank two semi-stable vector bundles with fixed determinant of degree d is known to be a rational projective variety (smooth if d is odd or $g = 2$) of dimension $3g - 3$. Its algebraic structure depends of D but as a topological space it is identified, if d is even, with the space of $SU(2)$ representations of $\pi_1(C)$. Let denote this space by $\mathcal{M}_g(d)$, assuming in what follows that $d = 0$ or 1 .

We will apply the construction of **Section 2** for obtaining for certain values of g a projective space \mathbb{P}^k embedded in $\mathcal{M}_g(d)$.

For this, let's denote by \mathcal{V} the complete variety constructed in **Section 2**. With the notations used there, let Z be a reduced 0-cycle on the curve $C \in |n_L H|$ formed by c_2 distinct points. Let $l \gg 0$ an integer and $D \in |lH|$ a smooth divisor which does not intersect Z . Let \mathcal{V}_Z the fiber over Z which is in fact $\mathbb{P}(\text{Ext}^1(LJ_Z, \mathcal{O}_S))$.

The main point is the following theorem which is a version of the **Theorem 1.1.** in [11] adapted for the presence of non locally free sheaves.

Theorem 4.1. *For $l \gg 0$ and $D \in |lH|$ as above, the restriction of every sheaf $E \in \mathcal{V}_Z$ on D is stable and the morphism*

$$\mathcal{V}_Z \rightarrow \mathcal{M}_D(d)$$

is an embedding, where $d = L \cdot D$.

We will give the sketch of the proof for the case we are interested in, where there are non locally free sheaves.

Proof: Let $M_H(2, L, c_2)$ be the moduli space of rank two H -stable vector bundles with the prescribed determinant and second Chern class. The result of Tyurin cited above asserts that for any c_2 there exists a $l_0(c_2)$ such that for any $l \geq l_0(c_2)$ and a generic smooth $D \in |lH|$ the restriction map

$$\bigcup M_H(2, L, k) \rightarrow \mathcal{M}_D(d)$$

is an embedding, where the union is taken for all $k \leq c_2$ and $d = L \cdot D$.

In what follows we will choose a l and a D as in the Tyurin theorem and such that $D \cap Z = \emptyset$. Therefore if $E \in \mathcal{V}_Z$ is a locally free sheaf, the restriction $E|_D$ is stable and, if $\mathcal{V}_{Z,lf}$ is the locus corresponding to locally free sheaves, then

$$\mathcal{V}_{Z,lf} \rightarrow \mathcal{M}_D(d)$$

is an embedding, by Tyurin theorem. We are therefore interested for the behaviour of the restriction map on the non locally free locus. Let E be such a sheaf. As we are on a surface, there is an exact sequence of the following type:

$$0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Ct(E) \rightarrow 0$$

where $E^{\vee\vee}$ is the bidual of E and $Ct(E)$ is the cotorsion sheaf. It is well known that the followings facts are true: $E^{\vee\vee}$ is locally free, H -stable if and only if E is H -stable, $c_2(E^{\vee\vee}) = c_2(E) - \text{length}(Ct(E))$, $Ct(E)$ is a sheaf with a 0-dimensional support included in the locus where E is not locally free therefore in Z , and E is isomorphic with $E^{\vee\vee}$ on $S \setminus Z$. So, as $D \cap Z = \emptyset$ we conclude that

$$E|_D \simeq E|_D^{\vee\vee}$$

But $E^{\vee\vee}$ is H -stable, locally free and from above $c_2(E^{\vee\vee}) < c_2(E)$. So by Tyurin theorem $E|_D^{\vee\vee}$ is stable. Moreover, for any $E_1 \in \mathcal{V}_Z$ and $E_1 \neq E$ we have $E_{1|D} \neq E|_D$ for the following reason: if E_1 is locally free we apply Tyurin theorem to E_1 and $E^{\vee\vee}$; if E_1 is not locally free we apply again Tyurin theorem to $E_1^{\vee\vee}$ and $E^{\vee\vee}$. QED

Remarque. The presence of such linear spaces in the moduli spaces of stable bundles over curves is not surprising in view of the rationality of these spaces proved in [9].

5 Relation with the spin strata

We recall the definition of spin strata from [10], [7] in the algebraic geometric context, without considering the differential geometric counterpart.

Let \mathcal{M} be the moduli space of rank two H -semistable sheaves with c_2 and determinant L fixed. The spin strata \mathcal{M}_i are defined as follows:

$$\mathcal{M}_i = \{E \in \mathcal{M} | H^0(E) + H^2(E) \geq i\}$$

. Moreover, the strata \mathcal{M}_i has a decomposition

$$\mathcal{M}_i = \bigcup \mathcal{M}_{j,l} ,$$

where the union is taken for all j, l such that $j + l = i$ and the $\mathcal{M}_{j,l}$ are defined by

$$\mathcal{M}_{j,l} = \{E \in \mathcal{M} | H^0(E) \geq j \text{ and } H^2(E) \geq l\}$$

. Using the construction in **Section 2** in the case $L \cdot H > 0$ we obtain the following corollary of **Theorem 2.2**:

Corollary 5.1. *If $L \cdot H > 0$ the spin stratum \mathcal{M}_1 contains a smooth complete subvariety \mathcal{V} of dimension $2c_2 + h^0(-L) - \chi(-L) - 1$.*

Moreover, if $H^2(\mathcal{O}_S) = H^2(L) = 0$, \mathcal{V} is not included in any higher codimensional stratum \mathcal{M}_i for $i \geq 2$ and in fact it is contained in $\mathcal{M}_{1,0}$.

Proof: The first assertion is obvious from the construction of the bundles $E \in \mathcal{V}$.

For the second we must show that $H^0(E) = 1$ and $H^2(E) = 0$ in the hypothesis of the Corollary. Using the sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow \mathcal{O}_S(L) \otimes \mathcal{I}_Z \rightarrow 0$$

we obtain that $H^2(E)$ is part of the following sequence:

$$\rightarrow H^2(\mathcal{O}_S) \rightarrow H^2(E) \rightarrow H^2(\mathcal{O}_S(L) \otimes \mathcal{J}_Z).$$

Now, the first term is 0 while the third is part of the sequence:

$$0 \rightarrow H^2(\mathcal{O}_S(L) \otimes \mathcal{J}_Z) \rightarrow H^2(L) \rightarrow 0$$

obtained by taking the cohomology of the following:

$$0 \rightarrow \mathcal{O}_S(L) \otimes \mathcal{J}_Z \rightarrow L \rightarrow \mathcal{O}_Z \rightarrow 0$$

So, using that $H^2(L) = 0$ we obtain $H^2(E) = 0$ as we wished.

Suppose now that $H^0(E) \geq 2$. We obtain a diagram of the following type:

$$\begin{array}{ccccccc} O & \longrightarrow & \mathcal{O}_S & \longrightarrow & E & \longrightarrow & \mathcal{O}_S(L) \otimes \mathcal{J}_Z \longrightarrow 0 \\ & & & & \uparrow & \nearrow \psi & \\ & & & & \mathcal{O}_S & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

and so ψ is a nonzero section in $\mathcal{O}_S(L) \otimes \mathcal{J}_Z$. But this contradicts one of the fundamental hypothesis of our construction concerning the choice of c_2 and of Z , namely:

$$c_2 > n_L \cdot L \cdot H.$$

So the Corollary is proved. QED

In the rest of this section we are concerned with the construction of complete subvarieties which are not strictly contained in any spin strata $\mathcal{M}_{j,l}$ with $j \geq 1$ or $l \geq 1$, and so we must modify the construction in **Section 2** for obtaining bundles without sections. For this let's making the following notation:

$$L' = L + 2mH$$

$$c'_2 = c_2 + mH \cdot L + m^2 H^2$$

where m is a positive integer. We want to apply the main construction in **Section 2** for the modified Chern classes L' and c'_2 . So we will obtain stable sheaves E' with the above Chern classes which are setting in exact sequences of the following type:

$$0 \rightarrow \mathcal{O}_S \rightarrow E' \rightarrow \mathcal{O}_S(L + 2mH) \otimes \mathcal{J}_{Z'} \rightarrow 0$$

where Z' is a zero-dimensional subscheme of length $l(Z') = c'_2$ such that the conditions in **Theorem 2.2.** are satisfied.

The bundles $E = E' \otimes \mathcal{O}_S(-mH)$ are obviously stable, as the E' are, and with Chern classes L and c_2 . So we obtain a new family \mathcal{V}'_m of stable bundles with the prescribed Chern classes.

The above remarks can be summarized in the following Corollary:

Corollary 5.2. *For $c_2 \gg 0$ the moduli space \mathcal{M} contains the complete subvariety \mathcal{V}'_m , the member of which are setting in sequences of the following type:*

$$0 \rightarrow \mathcal{O}_S(-mH) \rightarrow E \rightarrow \mathcal{O}_S(L + mH) \otimes \mathcal{J}_{Z'} \rightarrow 0$$

Remark. The explicit meaning of $c_2 \gg 0$ and the dimension of \mathcal{V}'_m can be obtained as in **Section 2** by making the computation for L' and c'_2 and then translating in terms of L and c_2 .

Another significant fact is that for fixed $c_2 \gg 0$ there are only a finite number of values for m such that the \mathcal{V}'_m can be constructed by the above procedure. This is a consequence of the fact that for $m \gg 0$ the growth of c'_2 is as $m^2 H^2$, while the growth of $\beta(L', H)$ is at last as $2m^2 H^2$ and the fundamental condition

$$c'_2 \geq \beta(L', H)$$

forbid that m would be too large.

Also, concerning the dimension of \mathcal{V}'_m , a simple computation shows that it is of the same order as the dimension of \mathcal{V} , that is of order $2c_2$. The topology of \mathcal{V}'_m is a little changed in that now it is a projective bundle over a symmetric power of order $c_2 + mH \cdot L + m^2 H^2$ over a curve in S . \square

We want now to see the relation of the \mathcal{V}'_m with the spin strata. First of all if $E \in \mathcal{V}'_m$ we have $H^0(E) = 0$. If not, we have a diagram:

$$\begin{array}{ccccccc}
O & \longrightarrow & \mathcal{O}_S(-mH) & \longrightarrow & E & \longrightarrow & \mathcal{O}_S(L+mH) \otimes \mathcal{J}_{Z'} \longrightarrow 0 \\
& & & & \uparrow & \nearrow \psi & \\
& & & & \mathcal{O}_S & & \\
& & & & \uparrow & & \\
& & & & 0 & &
\end{array}$$

with ψ obviously nonzero because $\mathcal{O}_S(-mH)$ has no sections. So ψ gives a nonzero section in $\mathcal{O}_S(L+mH) \otimes \mathcal{J}_{Z'}$ which is impossible by the choice of Z' and by the fundamental condition

$$c'_2 > n_{L'} \cdot L' \cdot H$$

used in the construction of the E' .

With the precedings notations we are now able to state the main result of this section:

Theorem 5.3. *If $L \cdot H > 0$ and $L \cdot H > K \cdot H$ then the complete subvariety \mathcal{V}'_m in \mathcal{M} is not strictly contained in any spin stratum $\mathcal{M}_{j,l}$ with $j \geq 1$ or $l \geq 1$. More exactly for $E \in \mathcal{V}'_m$ generic we have:*

$$H^0(E) = H^2(E) = 0$$

Proof: The conclusion for the H^0 follows for all the E in \mathcal{V}'_m by the precedings considerations.

In what follows consider E in \mathcal{V}'_m generic so that it is locally free and the zero-dimensional sub-scheme Z' which correspond to it consists of distinct reduced points. By taking the dual of the defining sequence of E and by tensoring it with $\mathcal{O}_S(K)$ we obtain the following:

$$0 \rightarrow \mathcal{O}_S(-L-mH+K) \rightarrow E^\vee \otimes \mathcal{O}_S(K) \rightarrow \mathcal{O}_S(K+mH) \otimes \mathcal{J}_{Z'} \rightarrow 0$$

By Serre duality we must show that $H^0(E^\vee \otimes \mathcal{O}_S(K)) = 0$. Suppose the contrary. We obtain the following diagram:

$$\begin{array}{ccccccc}
O & \longrightarrow & \mathcal{O}_S(-L-mH+K) & \longrightarrow & E^\vee \otimes \mathcal{O}_S(K) & \longrightarrow & \mathcal{O}_S(K+mH) \otimes \mathcal{J}_{Z'} \longrightarrow 0 \\
& & & & \uparrow \varphi & \nearrow \psi & \\
& & & & \mathcal{O}_S & & \\
& & & & \uparrow & & \\
& & & & 0 & &
\end{array}$$

Now, if $\psi = 0$ we obtain a section in $\mathcal{O}_S(-L - mH + K)$ which is impossible because the hypothesis $L \cdot H > K \cdot H$ would imply

$$(-L - mH + K) \cdot H < 0$$

So ψ is not zero and it defines a section in $\mathcal{O}_S(K + mH) \otimes \mathcal{J}_{Z'}$. But in the fundamental construction we have

$$c'_2 = \text{length}(Z') > n_{L'} \cdot L' \cdot H$$

and so Z' can not lying on a curve in the linear system $|mH + K|$ because the hypothesis $L \cdot H > K \cdot H$ imply

$$(mH + K) \cdot (n_{L'}H) < (mH + L) \cdot (n_{L'}H) < (L') \cdot (n_{L'}H)$$

So for E generic in \mathcal{V}'_m we have $H^2(E) = 0$ and the theorem is proved.
QED

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